The geometrical structure of a complexified theory of gravitation

1981 J. Phys. A: Math. Gen. 14 847

(http://iopscience.iop.org/0305-4470/14/4/014)

J. Phys. A: Matm. Gen. 14 (1981) 847–854. Printed in Great Britain

The geometrical structure of a complexified theory of gravitation

G Kunstatter† and R Yates‡

Blackett Laboratory, Imperial College of Science and Technology, Prince Consort Road, London SW7 2BZ, England

Received 30 June 1980

Abstract. We outline the procedure for the complexification of the tangent bundle over a four-dimensional space–time manifold. By introducing a connection and metric compatible with the complex structure, we form the geometrical basis for a new (complexified) theory of gravitation whose fundamental gauge group is U(3, 1). We further prove that the Lagrangian for the theory is necessarily real when the connection is compatible with the metric.

1. Introduction

Much effort has gone into trying to find a single theory which combines gravity and quantum mechanics in a self-consistent and compelling way. Since the field of complex numbers plays a fundamental role in quantum mechanics, it is natural to ask how one might include complex numbers into a geometrical description of physical space–time. The approach to be presented here is perhaps the simplest and most naive. The resulting (classical) theory, however, has many fascinating consequences which indicate that this approach might have brought us a step closer to the unification of gravitation and quantum mechanics.

The geometrical structure we shall use involves the complexification of the tangent bundle over a real four-dimensional manifold. Note that space–time itself is not complexified, in keeping with the experimental observation that physics seems to occur in an arena which is fundamentally four-dimensional and real. Section 2 of this paper describes the complexification procedure and introduces the dynamical structure required by a geometrical theory. Section 3 derives a Lagrangian from this structure and proves that it is pure real. This Lagrangian is related to the one used to derive a new non-symmetric theory of gravitation (Moffat 1979). The reality of this Lagrangian is a new result which emerges naturally from the formalism presented here. Finally, in § 4 we present conclusions and possible areas for future investigation.

2. The geometrical structure

We take M to be a four-dimensional differentiable manifold, labelled by real coordinates $\{x^\mu\}$, $\mu = 1, \ldots, 4$. The field of complex numbers is introduced simply by

† Supported in part by the Natural Sciences and Engineering Research Council of Canada.
‡ Supported by the Science Research Council of Great Britain under grant No NG 06586.

extending the algebra of real-valued functions on M to $\mathscr{F}^{\mathbb{C}}$, the algebra of complex-valued functions. A real representation for the complex-valued function $f^{\mathbb{C}}(x)$ is provided by the ordered pair of real-valued functions $(f^{\mathrm{R}}(x), f^{\mathrm{I}}(x))$, with canonical complex structure, J, such that†

$$J(f^{\mathrm{R}}, f^{\mathrm{I}}) = (-f^{\mathrm{I}}, f^{\mathrm{R}}). \tag{1}$$

It is this real representation on which we shall focus our attention, for reasons which will be made apparent.

Once complex-valued functions are given, it is natural to define a complex-valued vector, $A^{\mathbb{C}}$, which maps the algebra of complex-valued functions onto itself:

$$\begin{aligned} &A^{\mathbb{C}}: \mathscr{F}^{\mathbb{C}} \to \mathscr{F}^{\mathbb{C}} \\ &f^{\mathbb{C}}(x) \rightsquigarrow (A^{\mathbb{C}} f^{\mathbb{C}})(x). \end{aligned} \tag{2}$$

We choose to work with a real representation of $A^{\mathbb{C}}$, which we shall call $\tilde{A}$. We therefore define at each point $x \in M$ an eight-dimensional real vector space‡

$$\tilde{T}_x := T_x \times T_x, \tag{3}$$

where T_x is the tangent space of M at x. In general the entire group GL$(8, R)$ can act on the elements of $\tilde{T}_x$, so that patching all $(\tilde{T}_x \times U)$, where U is some neighbourhood of M, yields an associated fibre bundle with structure group GL$(8, R)$ and typical fibre $\tilde{T}_x \sim R^8$. Technically, we take the frame bundle $L(M)$ and form the associated bundle $L(M) \times_{\mathrm{GL}(4,R)} \mathrm{GL}(8, R)$. This is, the space consisting of the equivalence classes $[u, g] = \{(\omega, k)$ such that $\omega \in L(M)$, $k \in \mathrm{GL}(8, R)$, $\omega = uh$, $k = h^{-1}g$, $u \in L(M)$, $g \in \mathrm{GL}(8, R)$, $h \in \mathrm{GL}(4, R)\}$§.

We now define a complex structure J on $\tilde{T}_x$. Such complex structures are in one-to-one correspondence with reductions of the structure group GL$(8, R)$ to GL$(4, \mathbb{C})^3$ (Kobayashi and Nomizu 1964b). By defining J we reduce the bundle of linear frames‖ of $\tilde{T}_x$ to that of complex linear frames. We can always find a basis for $\tilde{T}_x$:

$$\{\tilde{e}_A\} := \{\tilde{e}_\alpha, \tilde{e}_{\bar{\alpha}}\}, \qquad A = 1, \ldots, 8, \qquad \alpha = 1, \ldots, 4, \qquad \bar{\alpha} := \alpha + 4 \tag{4a}$$

such that

$$\tilde{e}_{\bar{\alpha}} = J\tilde{e}_\alpha \tag{4b}$$

and hence

$$J\tilde{e}_{\bar{\alpha}} = -\tilde{e}_\alpha. \tag{4c}$$

In this basis J takes the form

$$J^{\bar{\alpha}}_{\beta} = \delta^\alpha{}_\beta, \qquad J^{\alpha}_{\bar{\beta}} = -\delta^\alpha{}_\beta, \qquad J^\alpha_\beta = J^{\bar{\alpha}}_{\bar{\beta}} = 0. \tag{5a, 5b, 5c}$$

This basis allows the canonical choice of a four-dimensional subspace, namely that subspace spanned by $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$, which may be identified with T_x. $\tilde{T}_x$ now provides a real representation for complex-valued vectors, with the following correspondence:

$$\tilde{A}(x) \to A(x) = (\tilde{A}^\alpha(x) + \mathrm{i}\tilde{A}^{\bar{\alpha}}(x))e_\alpha \tag{6}$$

† Arguments of functions will henceforth be suppressed where no ambiguity results.

‡ Actually $\tilde{T}_x$ is the space of ordered pairs (A, B) of vectors $A, B \in T_x$, with the multiplication law $\lambda(A, B) = (\lambda A, \lambda B)$, and the addition law $(A', B') + (A, B) = (A' + A, B' + B)$.

§ See Kobayashi and Nomizu (1964a, p 55) for details.

‖ A linear frame at x is defined to be a non-singular linear map from $R^8 \to \tilde{T}_x$.

where $\{e_\alpha\}$ is a basis for T_x. Since $\tilde{T}_x = T_x \times T_x$, we have chosen a basis with $\tilde{e}_\alpha = (e_\alpha, 0)$ and $\tilde{e}_{\bar{\alpha}} = (0, e_\alpha)$. Hence each vector $\tilde{A} \in \tilde{T}_x$ represents an ordered pair $(A^{\mathrm{R}}(x), A^{\mathrm{I}}(x))$ of 4-vectors in T_x, such that $A^{\mathrm{R}}(x)$ is the 'real part' of $\tilde{A}(x)$ and $A^{\mathrm{I}}(x)$ is the 'imaginary part' of $\tilde{A}(x)$. One important consequence of this choice is that coordinate transformations of M induce the real subgroup of GL(4, $\mathbb{C}$) transformations on $\tilde{T}_x$. It is important to note that no dynamical structure has yet been introduced on M, since the complex structures is fixed *a priori*.

It is now possible to define the action of a complex-valued vector $A^{\mathbb{C}}$ on a complex-valued function $f^{\mathbb{C}}$, in terms of their real representations. In particular

$$\tilde{A}:(f^{\mathrm{R}}, f^{\mathrm{I}}) \mapsto (A^{\mathrm{R}}f^{\mathrm{R}} - A^{\mathrm{I}}f^{\mathrm{I}}, A^{\mathrm{R}}f^{\mathrm{I}} + A^{\mathrm{I}}f^{\mathrm{R}}), \tag{7}$$

where $A^{\mathrm{R}}f^{\mathrm{R}}, A^{\mathrm{I}}f^{\mathrm{I}}$ etc are defined in the usual way since $A^{\mathrm{R}}, A^{\mathrm{I}}$ and $f^{\mathrm{R}}, f^{\mathrm{I}}$ are real vectors and functions respectively. Equation (7) leads to the unique definition of the Lie bracket $[\tilde{A}, \tilde{B}]$ on $\tilde{T}_x$

$$[\tilde{A}, \tilde{B}] := ([A^{\mathrm{R}}, B^{\mathrm{R}}] - [A^{\mathrm{I}}, B^{\mathrm{I}}], [A^{\mathrm{R}}, B^{\mathrm{I}}] + [A^{\mathrm{I}}, B^{\mathrm{R}}]) \tag{8}$$

It is interesting to note that with the above definitions the torsion of the complex structure

$$N_J(\tilde{A}, \tilde{B}) := \{[J\tilde{A}, J\tilde{B}] - [\tilde{A}, \tilde{B}] - J[\tilde{A}, J\tilde{B}] - J[J\tilde{A}, \tilde{B}]\} \tag{9}$$

is identically zero.

We now have an associated GL(4, $\mathbb{C}$) bundle over M, with typical fibre $\tilde{T}_x \sim \mathbb{C}^4$. It is possible to define a connection over the bundle, which tells us how to move horizontally from one fibre, $\tilde{T}_x$, to another, $\tilde{T}_{x'}$. Since $(x'-x)$ is locally determined by a vector in T_x, our connection defines the covariant derivative operator, ∇, which is a map from $T_x \times \tilde{T}_x$ into $\tilde{T}_x$. In particular, for $\tilde{A} \in \tilde{T}_x$, $B \in T_x$, we have

$$\nabla_B \tilde{A} = \tilde{A}^A(x)\nabla_B \tilde{e}_A + (B\tilde{A}^A(x))\tilde{e}_A. \tag{10}$$

Since $\tilde{A}^A(x)$ is a real function on M and B is a real vector on M, $B\tilde{A}^A(x)$ is well defined. In a coordinated basis we have

$$\nabla_B \tilde{A} = [B^\mu \tilde{A}^A \Gamma^C_{\mu A} + B^\mu(\partial \tilde{A}^C/\partial x^\mu)]\tilde{e}_C, \tag{11}$$

where we have defined

$$\Gamma^C_{\mu A}\tilde{e}_C := \nabla_{e_\mu}\tilde{e}_A. \tag{12}$$

Note that the indices referring to the basis $\{\tilde{e}_A\}$ of $\tilde{T}_x$ are *internal* indices, and must be kept distinct from the space–time indices referring to $\{e_\mu\}$ of T_x. One of the advantages of using the real representation of the complexified tangent space is that this distinction is manifest.

So far the components $\Gamma^A_{\mu B}$, denote the 4×8^2 degrees of freedom of a GL(8, R) connection over M. In order to restrict ourselves to the desired GL(4, $\mathbb{C}$) connection, we require the connection to be compatible with the complex structure. The condition is

$$(\nabla J) = 0, \tag{13a}$$

or

$$\nabla(J\tilde{A}) = J(\nabla \tilde{A}), \tag{13b}$$

which yields via equation (5)

$$\Gamma^{\alpha}_{\mu\beta} = \Gamma^{\bar{\alpha}}_{\mu\bar{\beta}}, \tag{14a}$$

$$\Gamma^{\alpha}_{\mu\bar{\beta}} = -\Gamma^{\bar{\alpha}}_{\mu\beta}. \tag{14b}$$

We are thus left with the desired 2×4^3 independent degrees of freedom of a GL(4, $\mathbb{C}$) connection over M. A complex-valued connection $W^{\mathbb{C}}$ can now be represented by

$$W^{\mathbb{C}\alpha}_{\mu\beta} := \Gamma^{\alpha}_{\mu\beta} + \mathrm{i}\Gamma^{\bar{\alpha}}_{\mu\beta}. \tag{15}$$

As in general relativity, we now introduce a metric structure $\tilde{g}(x)$ on each fibre $\tilde{T}_x$, with the added condition that it be compatible with the complex structure. In other words $\tilde{g}$ is a *Hermitian* fibre metric:

$$\tilde{g}(\tilde{A}, \tilde{B}) = \tilde{g}(\tilde{B}, \tilde{A}) = \tilde{g}(J\tilde{A}, J\tilde{B}) = \tilde{g}_{AB}\tilde{A}^A\tilde{B}^B. \tag{16}$$

Note that

$$\tilde{g}_{AB} = \tilde{g}_{BA} \tag{17a}$$

while

$$\tilde{g}_{\bar{\alpha}\bar{\beta}} = \tilde{g}_{\alpha\beta} \tag{17b}$$

$$\tilde{g}_{\alpha\bar{\beta}} = -\tilde{g}_{\beta\bar{\alpha}}. \tag{17c}$$

$\tilde{g}$ provides a real representation for a *sesquilinear* Hermitian form $g^{\mathbb{C}}$ acting on complex-valued vectors as follows:

$$g^{\mathbb{C}}(A^{\mathbb{C}}, B^{\mathbb{C}}) := \tilde{g}(\tilde{A}, \tilde{B}) + \mathrm{i}\tilde{g}(\tilde{A}, J\tilde{B}), \tag{18a}$$

so that

$$g^{\mathbb{C}}_{\mu\nu} = \tilde{g}_{\mu\nu} + \mathrm{i}\tilde{g}_{\mu\bar{\nu}}. \tag{18b}$$

We now consider the condition necessary for the connection defined in equation (5) to be compatible with the metric structure. The condition is

$$\nabla\tilde{g} = 0, \tag{19a}$$

or

$$\nabla_A(\tilde{g}(\tilde{B}, \tilde{C})) = \tilde{g}(\nabla_A\tilde{B}, \tilde{C}) + \tilde{g}(\tilde{B}, \nabla_A\tilde{C}); \tag{19b}$$

which implies that

$$\tilde{g}_{AB,\mu} - \Gamma^{C}_{\mu A}\tilde{g}_{CB} - \Gamma^{C}_{\mu B}\tilde{g}_{AC} = 0 \tag{19c}$$

in a coordinated basis for T_x. In terms of the complex-valued metric and the connection of equations (18a) and (15), equation (19c) is just the familiar condition (Kunstatter 1979, 1980)

$$g^{\mathbb{C}}_{\mu\nu,\lambda} - g^{\mathbb{C}}_{\epsilon\nu}W^{\mathbb{C}\epsilon}_{\lambda\mu} - g^{\mathbb{C}}_{\mu\epsilon}\overline{W^{\mathbb{C}\epsilon}_{\lambda\nu}} = 0. \tag{20}$$

By introducing a metric and connection on $\tilde{T}_x$ which are compatible with each other and with the complex structure, we have reduced the structure group of our fibre bundle from GL(4, $\mathbb{C}$) to U(3, 1). At each point x we can find eight orthonormal frames $\{\tilde{h}_I\} = \{\tilde{h}_i, J\tilde{h}_i\}$; $i, j = 1, \ldots, 4$; $I, J = 1, \ldots, 8$, which remain orthogonal under unitary transformations. That is

$$\tilde{g}(\tilde{h}_I, \tilde{h}_J) = \eta_{IJ} = \tilde{g}(\Lambda^K_I\tilde{h}^I_K, \Lambda^L_J\tilde{h}_L), \tag{21}$$

where $\eta_{IJ} = \begin{pmatrix} \eta_{ij} & 0 \\ 0 & \eta_{ij} \end{pmatrix}$ and Λ_I^K is an element of the eight-dimensional real representation of the group U(3, 1). The $\tilde{h}_I$ provide a real representation for complex-valued tetrads over M:

$$\tilde{h}_i^{\mathbb{C}} := (\tilde{h}_i^{\mu} + i\tilde{h}_i^{\bar{\mu}})e_{\mu}. \tag{22}$$

The $\tilde{h}_{\bar{i}} = J\tilde{h}_i$ provide no new information since

$$\tilde{h}_{\bar{i}}^{\mu} = -\tilde{h}_i^{\bar{\mu}} \tag{23}$$

$$\tilde{h}_{\bar{i}}^{\bar{\mu}} = \tilde{h}_i^{\mu}. \tag{24}$$

The entire structure described above has been formulated in an elegant way using G structures by Coleman (1980). While his formalism is more compact, we feel that the present formulation makes explicit the true geometrical nature of the complex-valued vectors and forms which are used. The metric $g^{\mathbb{C}}$ is defined totally in terms of a real, symmetric metric on $\tilde{T}_x$. The 'ungeometrical' skew parts of $g^{\mathbb{C}}$ are only a manifestation of the complex-valued representation. In addition, as previously stated, by working with the real representation, one is forced to distinguish between the internal degrees of freedom reflected by the complexification of the tangent space, and the degrees of freedom allowed by translations in *real* four-dimensional space–time.

3. The Lagrangian

In order to formulate a physical theory, we must construct a Lagrangian $\mathscr{L}(x)$ from the geometrical structures at hand, namely J, ∇ and $\tilde{g}$ (or equivalently $\{\tilde{h}_I\}$). To this end we first define the curvature form $R(A, B)$, which, given two vectors $A, B \in T_x$, maps $\tilde{T}_x$ onto itself:

$$R(A, B): \tilde{C} \rightsquigarrow R(A, B)\tilde{C} = (\nabla_A\nabla_B - \nabla_B\nabla_A - \nabla_{[A,B]})\tilde{C} = \tilde{C}^B A^{\mu} B^{\nu} R^{A}_{\mu\nu B}\tilde{e}_A, \tag{25}$$

where, in a coordinated basis,

$$R^{A}_{\mu\nu B} = \Gamma^{A}_{\mu B,\nu} - \Gamma^{A}_{\nu B,\mu} - \Gamma^{A}_{\mu C}\Gamma^{C}_{\nu B} + \Gamma^{A}_{\nu C}\Gamma^{C}_{\mu B}. \tag{26}$$

It can easily be verified that equations (14a) and (14b) imply that

$$R^{\alpha}_{\mu\nu\beta} = R^{\bar{\alpha}}_{\mu\nu\bar{\beta}}, \tag{27a}$$

$$R^{\alpha}_{\mu\nu\bar{\beta}} = -R^{\bar{\alpha}}_{\mu\nu\beta}. \tag{27b}$$

We therefore have the complex-valued curvature tensor

$$\begin{aligned} R^{\mathbb{C}\alpha}_{\mu\nu\beta} &:= R^{\alpha}_{\mu\nu\beta} + \mathrm{i}R^{\bar{\alpha}}_{\mu\nu\beta} \\ &= W^{\mathbb{C}\alpha}_{\mu\beta,\nu} - W^{\mathbb{C}\alpha}_{\nu\beta,\mu} - W^{\mathbb{C}\alpha}_{\mu\epsilon}W^{\mathbb{C}\epsilon}_{\nu\beta} + W^{\mathbb{C}\alpha}_{\nu\epsilon}W^{\mathbb{C}\epsilon}_{\mu\beta}. \end{aligned} \tag{28}$$

$R^{A}_{\mu\nu B}\,\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu}$ defines a two-form in $T_x^*\otimes T_x^*$. It can, however, be lifted to $\tilde{T}_x^*\otimes\tilde{T}_x^*$ by defining

$$\tilde{R}(\tilde{A}, \tilde{B}) := R(A^{\mathrm{R}}, B^{\mathrm{R}}) \tag{29}$$

where $A^{\mathrm{R}}, B^{\mathrm{R}}$ are the real projections of $\tilde{A}, \tilde{B}$ onto T_x. Although $\tilde{R}(\tilde{A}, \tilde{B})$ is not invariant under general unitary transformations, it is invariant under the subgroup of real transformations induced by coordinate transformations of M.

There are only four independent scalars which are linear in $\tilde{R}$. They are as follows:

(i) $$\tilde{g}(\tilde{R}(\tilde{h}_I, \tilde{h}_J)\tilde{h}_K, \tilde{h}_L)\eta^{IK}\eta^{JL} = \tilde{g}^{\mu B}R^{\ \nu}_{\mu\nu B} \tag{30a}$$

(ii) $$\tilde{g}(\tilde{R}(\tilde{h}_I, \tilde{h}_J)\tilde{h}_K, J\tilde{h}_L)\eta^{IK}\eta^{JL} = \tilde{g}^{\mu B}R^{\ \bar{\nu}}_{\mu\nu B} \tag{30b}$$

(iii) $$\tilde{g}(\tilde{R}(\tilde{h}_I, J\tilde{h}_J)\tilde{h}_K, \tilde{h}_L)\eta^{IJ}\eta^{KL} = \tilde{g}^{\mu\nu}R^{\ A}_{\mu\nu A} \tag{30c}$$

(iv) $$\tilde{g}(\tilde{R}(\tilde{h}_I, J\tilde{h}_J)\tilde{h}_K, J\tilde{h}_L)\eta^{IJ}\eta^{KL} = \tilde{g}^{\mu\nu}R^{\ \bar{\rho}}_{\mu\nu\rho}. \tag{30d}$$

A sample derivation of the component form of these quantities is given in the Appendix. Note that these scalars are also invariant under unitary transformations of the frames: $\tilde{h}_I \to \tilde{h}'_I = \Lambda^J_I\tilde{h}_J$, because all internal tetrad indices have been contracted out.

The meaning of the terms (i)–(iv) is apparent when they are written in terms of $g^{\mathbb{C}}$ and $R^{\mathbb{C}}$:

(i) $$\mathrm{Re}(g^{\mathbb{C}\mu\nu}R^{\mathbb{C}\alpha}_{\ \mu\alpha\nu}) \tag{31a}$$

(ii) $$\mathrm{Im}(g^{\mathbb{C}\mu\nu}R^{\mathbb{C}\alpha}_{\ \mu\alpha\nu}) \tag{31b}$$

(iii) $$\mathrm{Re}(g^{\mathbb{C}\mu\nu}R^{\mathbb{C}\alpha}_{\ \mu\nu\alpha}) \tag{31c}$$

(iv) $$\mathrm{Im}(g^{\mathbb{C}\mu\nu}R^{\mathbb{C}\alpha}_{\ \mu\nu\alpha}) \tag{31d}$$

where Re and Im denote real and imaginary parts respectively. Thus, in terms of the complex-valued tensors, the (complex) Lagrangian is merely a linear combination of the traces of the first and second contractions of the generalised curvature tensor. Moreover, it is shown in the Appendix that when the connection is metrically compatible, both (ii) and (iv) are *identically zero.* We therefore have the important result that the geometrical structure guarantees the reality of the Lagrangian.

Of course, in order to perform an integration over M, we require a scalar density. The most natural choice is simply $(-\det g^{\mathbb{C}})^{1/2} = (-\det \tilde{g})^{1/4}$, which is necessarily real when $g^{\mathbb{C}}$ is Hermitian and non-degenerate. The Lagrangian density, in terms of the complex-valued tensors, is therefore

$$\mathscr{L}(x) = (-g^{\mathbb{C}})^{1/2}(ag^{\mathbb{C}\mu\nu}R^{\mathbb{C}\alpha}_{\ \mu\nu\alpha} + bg^{\mathbb{C}\mu\nu}R^{\mathbb{C}\alpha}_{\ \mu\alpha\nu}) \tag{32}$$

where a and b are arbitrary real parameters.

4. Conclusions

We have examined the geometrical structure which results when the algebra of real functions over space–time is extended to that of complex-valued functions. We have also constructed a gauge-invariant Lagrangian and shown it to be real, even when the geometrical fields are complex-valued. This Lagrangian is in fact related to the one used to derive the vacuum field equations for a new theory of gravitation which has recently been the subject of much research (Moffat 1979, Kunstatter 1979).

By focusing on the real representation for the complex structure, the difference between internal indices and space–time indices is made manifest. It is in fact possible to formulate the above structure in terms of a fully complex base manifold. The extra degrees of freedom which occur must then be removed by appropriate constraints on the field variables. This is in fact analogous to the superspace formulation of supergravity, in which some form of dimensional reduction (Brink *et al* 1977, Sohnius *et al* 1980) is needed to regain four-dimensional space–time.

It is well known that complex numbers and unitary gauge groups play an important role in modern physics. The present paper has described one way in which this complex structure might interact with the geometrical structure of space–time. Much work remains to be done before the full implications of this approach are known. One possible area for future investigation lies in the further generalisation of complex-valued functions on the manifold to functions which map M into R^n, for example. It would also be interesting to ask what would happen if the complex structure were treated as a dynamical field, instead of being fixed *a priori*. More important, however, is the investigation of the physical implications of a theory of gravitation based on the structure presented here, so that one might check experimentally whether or not it describes the geometry of our physical world.

Appendix

We shall now derive explicitly equation (30b).

$$\tilde{g}(\tilde{R}(\tilde{h}_I, \tilde{h}_J)\tilde{h}_K, J\tilde{h}_L)\eta^{IK}\eta^{JL} = \tilde{h}_I^\mu \tilde{h}_J^\nu R^A_{\mu\nu B}\tilde{h}_K^B (J\tilde{h}_L)^C \tilde{g}_{AC}\eta^{IK}\eta^{JL}. \tag{A1}$$

But from equation (21) it can be shown that

$$\tilde{h}_I^A \tilde{h}_J^B \eta^{IJ} = \tilde{g}^{AB}, \tag{A2}$$

where $\tilde{g}^{AB}$ is the inverse of $\tilde{g}_{AC}$:

$$\tilde{g}^{AB}\tilde{g}_{AC} = \delta^B_C. \tag{A3}$$

Moreover

$$(J\tilde{h}_L)^C = J^C_D \tilde{h}^D_L, \tag{A4}$$

so that equation (A1) yields

$$\text{(ii)} = \tilde{g}^{\mu B}\tilde{g}^{\nu D} J^C_D R^A_{\mu\nu B}\tilde{g}_{AC}. \tag{A5}$$

However, equation (5) implies that

$$\begin{aligned} \tilde{g}^{\nu D} J^C_D \tilde{g}_{AC} &= \tilde{g}^{\nu\alpha} J^C_\alpha \tilde{g}_{AC} + \tilde{g}^{\nu\bar{\alpha}} J^C_{\bar{\alpha}} \tilde{g}_{AC} \\ &= \tilde{g}^{\nu\alpha}\tilde{g}_{A\bar{\alpha}} - \tilde{g}^{\nu\bar{\alpha}}\tilde{g}_{A\alpha} \\ &= \tilde{g}^{\bar{\nu}\bar{\alpha}}\tilde{g}_{A\bar{\alpha}} + \tilde{g}^{\bar{\nu}\alpha}\tilde{g}_{A\alpha} \\ &= \delta^{\bar{\nu}}_A, \end{aligned} \tag{A6}$$

where we have used equations (17b) and (17c). Substitution of equation (A6) into (A5) then yields the result in equation (30b).

We shall now prove that the Lagrangian is pure real. This is done most simply in terms of the complex-valued tetrads $h_i^{\mathbb{C}}$ of equation (22) and the complex-valued curvature form $R_j^{\mathbb{C}i}$ which acts on the tetrads:

$$h_i^{\mathbb{C}} \to R_i^{\mathbb{C}j} h_j^{\mathbb{C}}. \tag{A7}$$

It is straightforward to show that the $\{h_i^{\mathbb{C}}\}$ will be horizontal only if $R^{\mathbb{C}ij} = -\overline{R^{\mathbb{C}ji}}$, where the bar denotes complex conjugation and we have raised the tetrad index with η^{ij}. That is, the curvature form must be anti-Hermitian. It can also be seen that (i) and (ii) of equations (30*a*) and (30*b*) are the real and imaginary parts, respectively, of

$$h_i^{\mathbb{C}\mu}\overline{h_j^{\mathbb{C}\bar{\nu}}}R_{\mu\nu}^{\mathbb{C}ij} = \overline{(\overline{h_i^{\mathbb{C}\mu}}h_j^{\mathbb{C}\nu}\overline{R_{\mu\nu}^{\mathbb{C}ij}})} = -\overline{(\overline{h_i^{\mathbb{C}\bar{\mu}}}h_j^{\mathbb{C}\nu}R_{\mu\nu}^{\mathbb{C}ji})} = +\overline{(\overline{h_j^{\mathbb{C}\bar{\nu}}}h_i^{\mathbb{C}\mu}R_{\mu\nu}^{\mathbb{C}ij})}, \qquad \text{(A8)}$$

where we have interchanged dummy variables and used the symmetries of $R_{\mu\nu}^{\mathbb{C}ij}$. Thus term (ii) must be zero.

Terms (iii) and (iv) in equations (30*a*) and (30*b*) correspond to the real and imaginary parts of

$$h_i^{\mathbb{C}\mu}\overline{h^{\mathbb{C}i\nu}}R_{j\mu\nu}^{\mathbb{C}j}. \qquad \text{(A9)}$$

But since $R^{\mathbb{C}ij}$ is anti-Hermitian, its trace must be imaginary. Hence, $R_{j\mu\nu}^{\mathbb{C}j}$ is Hermitian in the indices μ and ν. Equation (A9) is therefore real and term (iv) must vanish. This completes the proof.

Of course, we could have used the real representation or the complex-valued tensors $g_{\mu\nu}^{\mathbb{C}}$ and $R_{\pi\mu\nu}^{\mathbb{C}\mathbb{C}\lambda}$ in the above proof. However, the method presented is perhaps the briefest and most straightforward.

References

Brink L, Schwartz J and Scherk J 1977 *Nucl. Phys.* B **121** 253
Coleman R 1980 Preprint University of Toronto
Kobayashi S and Nomizu K 1964a *Foundations of Differential Geometry* vol 1 (New York: Wiley)
—— 1964b *Foundations of Differential Geometry* vol 2 (New York: Wiley)
Kunstatter G 1979 *PhD Thesis* University of Toronto
—— 1980 *Gen. Rel. Grav.* in press
Moffat J W 1979 *Phys. Rev.* D **19** 3554
Sohnius M F, Stelle K G and West P C 1980 *Europhysics Conf. Unification of the Fundamental Interactions (Erice, Sicily)*